B. Sc. (Physics) Part-III, Paper-V, Group-A

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Tensor

Before moving to the details of tensor, we review here some elementary physical laws which will give you some feelings of tensor uses in physics.

$$\vec{a} = \frac{\vec{F}}{m} \quad \text{--- (1)}$$ → acceleration of a body is proportional to the force acting on it

$$\vec{J} = \sigma \vec{E} \quad \text{--- (2)}$$ → Electric ~~field~~ current in a medium is proportional to the applied field.

The above physical laws are a special case and apply strictly only to isotropic media (a medium whose properties are same in all directions) or a media which posses high symmetry.

In case of anisotropic media acceleration $\vec{a}$ is not necessarily parallel to the applied force ~~and~~ (Eq. 1) Or the current flows in a direction different from that of the electric field (Eq. 2).

Eq. 1 or Eq. 2 for anisotropic media can be written in a generalized form. We take Eq. 2 —

$$J_x = \sigma_{xx} E_x + \sigma_{xy} E_y + \sigma_{xz} E_z$$

$$J_y = \sigma_{yx} E_x + \sigma_{yy} E_y + \sigma_{yz} E_z$$

$$J_z = \sigma_{zx} E_x + \sigma_{zy} E_y + \sigma_{zz} E_z$$

J_x, J_y, J_z and E_x, E_y, E_z are respectively the Cartesian components of $\vec{J}$ and $\vec{E}$, and σ_{ij} ($i, j = x, y, z$) are said to be the components of the conductivity tensor

Similar Eq ① can be generalized with $\left(\frac{1}{m}\right)_{ij}$ denoting the components of mass tensor (reciprocal mass tensor) of the particle in the medium.

Tensor Application — mainly in relativistic physics → special theory of relativity, general theory of relativity etc

Conventions & Notations:

Consider an N-dimensional space and let $x^1, x^2, x^3, \ldots, x^N$ be ~~the~~ any set of coordinate in this space.

Note that here ~~in x^i is the~~ in writing x^i, i is the superscript on x not the ith power on x.

When it will be needed to write power on x^i we will write it like $(x^i)^2$, $(x^i)^3$, etc

N-dimensional space under the consideration will be denoted by V_n.

A notation $f \equiv f(t)$ will mean that f is a function of t.

Let $\bar{x}^\alpha$ ($1 \leq \alpha \leq N$) be another set of coordinates in the same space V_n. Each of the coordinates x^i will be a function of the N coordinates $\bar{x}^\alpha$ and vice versa.

Therefore, we can write

$$x^i = x^i(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^N), \qquad 1 \leq i \leq N \quad \text{—— (3)}$$

$$\bar{x}^\alpha = \bar{x}^\alpha(x^1, x^2, \ldots, x^N), \qquad 1 \leq \alpha \leq N \quad \text{—— (4)}$$

Example: Express the Cartesian and the spherical polar coordinates as functions of each other.

Soln: We have already discussed curvilinear coordinates in earlier lecture note. We use ~~some~~ those concepts here.

Let x, y, z denote the Cartesian coordinates and r, θ, ϕ the spherical polar coordinates in a three-dimensional space. The coordinates x, y, z are independent of each other. Similarly, r, θ, ϕ are independent of each other.

The coordinates of one set are functions of those of the other set. The Cartesian coordinates are related to the spherical polar coordinates by

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

The inverse transformation is given by

$$r = (x^2+y^2+z^2)^{\frac{1}{2}}, \quad \theta = \tan^{-1}\left[\frac{\sqrt{x^2+y^2}}{z}\right], \quad \phi = \tan^{-1}(y/x)$$

You see that (r, θ, ϕ) can be expressed as functions of (x, y, z), and vice versa.

Now differentiating Equations (3) and (4) we can write.

$$dx^i = \sum_{\alpha=1}^{N} \frac{\partial x^i}{\partial \bar{x}^\alpha} d\bar{x}^\alpha, \quad 1 \leq i \leq N \qquad (6)$$

$$d\bar{x}^\alpha = \sum_{i=1}^{N} \frac{\partial \bar{x}^\alpha}{\partial x^i} dx^i, \quad 1 \leq \alpha \leq N. \qquad (7)$$

If we use Einstein's summation convention, it will simplify the above notations.

Einstein's Summation convention: If an index (except N) is repeated in a term, summation over it from 1 to N is implied.

Therefore, we can write

$$dx^i = \frac{\partial x^i}{\partial \bar{x}^\alpha} d\bar{x}^\alpha , \quad 1 \le i \le N \quad \text{--- (8)}$$

$$d\bar{x}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^i} dx^i , \quad 1 \le \alpha \le N \quad \text{--- (9)}$$

Note that α, appears twice on the rhs of Eq. (8) and i appears twice on the rhs of Eq. (9), therefore, summation over these from 1 to N is applied in the respective Equations.

We use this convention throughout this whole discussion of tensor analysis.

It is to be noted that if an index appears only once in any term, it has a definite value (any value from 1 to N). This index is called as free index. In Eq. (8), i is free index and in Eq. (9), α is free index. Further, we drop $1 \le i \le N$ in Eq. (8) and $1 \le \alpha \le N$ in Eq. (9). this should be understood.

free index

$$dx^i = \frac{\partial x^i}{\partial \bar{x}^\alpha} d\bar{x}^\alpha$$

dummy index

$$d\bar{x}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^i} dx^i$$

Dummy index → An index which is repeated and over which summation is implied is called a dummy index. Dummy index can be replaced by any other index which does not appear in the same term.

~~Let a_i, b_i, c_i, d_i, $1 \le i \le N$ be four~~

Tensor continued...

In the last lecture note, we have discussed basic idea about tensor, ~~an~~ its conventions & notations which we will follow in whole discussion of tensor analysis. We have also discussed Einstein's summation convention, dummy index, free index.

We have obtained relation (see earlier note)

$$dx^{\nu} = \frac{\partial x^{\nu}}{\partial \bar{x}^{\alpha}} d\bar{x}^{\alpha} \qquad \text{— (1)}$$

free index — ν; dummy index — α

$$d\bar{x}^{\alpha} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{\nu}} dx^{\nu} \qquad \text{— (2)}$$

free index — α; dummy index — ν

Let's discuss an example to clarify more about Einstein's summation convention.

Ex. Let a_i, b_i, c_i, d_i, $1 \leq i \leq N$, be four sets of N quantities each. Then according to the Einstein's summation convention, we have

$$a_i b_i \equiv a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots + a_N b_N \qquad \text{— (3)}$$

and

$$a_i b_j c_j \equiv a_i b_1 c_1 + a_i b_2 c_2 + a_i b_3 c_3 + \cdots + a_i b_N c_N \qquad \text{— (4)}$$

i is free index here ~~[illegible]~~ have fixed value between 1 to N.

Eq. (3) can also be written as

$$a_i b_i = a_j b_j = a_k b_k = a_r b_r = a_l b_l \text{, etc} \quad (5)$$

In above equation same index is occurring twice in a term. These are dummy indices.

Again

$$a_i b_j c_j = a_i b_k c_k = a_i b_l c_l \quad (6)$$

~~i is dummy index in above equation~~

j, k, l are dummy indices in above ~~equation~~ expression which cannot be replaced by 'i' since 'i' appears in the same term.

Therefore, $a_i b_j c_j \neq a_i b_i c_i$ — (7)

The above Eq. can be verified, write

$$a_i b_i c_i \equiv a_1 b_1 c_1 + a_2 b_2 c_2 + a_3 b_3 c_3 + \cdots + a_N b_N c_N \quad (8)$$

See Eq. (8) and (4) are not same. Thus, Eq. (7) is true.

Again consider expressions $a_i b_i c_i d_i$ and $a_i b_i c_j d_j$.

Now,

$$a_i b_i c_i d_i \equiv a_1 b_1 c_1 d_1 + a_2 b_2 c_2 d_2 + a_3 b_3 c_3 d_3 + \cdots + a_N b_N c_N d_N; \quad (9)$$

$$a_i b_i c_j d_j \equiv \left(\sum_{i=1}^{N} a_i b_i\right)\left(\sum_{j=1}^{N} c_j d_j\right)$$

$$= (a_1 b_1 + a_2 b_2 + \cdots + a_N b_N)(c_1 d_1 + c_2 d_2 + \cdots + c_N d_N) \quad (10)$$

From Eqs. (9) and (10)

$$a_i b_i c_i d_i \neq a_i b_i c_j d_j$$

Also we can write $a_i b_i c_j d_j = a_i b_i c_k d_k = a_l b_l c_i d_i$, etc

~~Consider Eq. ① and ②, From Eq. ① we can write~~

Since coordinates x^i are independent of each other, therefore

$$\frac{dx^i}{dx^j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases} \qquad \text{—— (11)}$$

We define the Kronecker delta symbol by

$$\delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad \text{—— (12)}$$

Now Eq. (11) ~~and (12)~~ can be written as

$$\frac{dx^i}{dx^j} = \delta^i_j, \qquad \text{—— (13)}$$

Similarly, the coordinates $\bar{x}^\alpha$ are also independent of each other, so that

$$\frac{d\bar{x}^\alpha}{d\bar{x}^\beta} = \delta^\alpha_\beta$$

If x^i are functions of $\bar{x}^\alpha$, then we can write

$$\frac{dx^i}{dx^j} = \frac{\partial x^i}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^\alpha}{\partial x^j}$$

using Eq. (11) and (12), we can write.

$$\frac{\partial x^i}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^\alpha}{\partial x^j} = \delta^i_j \qquad \text{—— (14)}$$

Similarly, we obtain

$$\frac{\partial \bar{x}^\alpha}{\partial x^k} \frac{\partial x^k}{\partial \bar{x}^\beta} = \delta^\alpha_\beta \qquad \text{—— (15)}$$

Contravariant vector:

Tensors are defined using the properties of their transformation rules under coordinate transformations. Vectors are the special case of tensors.

Consider a physial enity is characterized by N functions A^i when expressed in the x^i coordinate system.

Let the same enity be characterized by $\bar{A}^\alpha$ when it is measured in coordinate system $\bar{x}^\alpha$.

A^i are called components of a <u>contravariant vector</u> if they transform under coordinates transformations given as

$$\boxed{\bar{A}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^j} A^j}, \quad \text{--- (1)}$$

which can be inverted to obtain A^i in terms of $\bar{A}^\alpha$

multiplying (1) by $\frac{\partial x^k}{\partial \bar{x}^\alpha}$ and summing over all α.

$$\frac{\partial x^k}{\partial \bar{x}^\alpha} \bar{A}^\alpha = \frac{\partial x^k}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^\alpha}{\partial x^j} A^j$$

$$\frac{\partial x^k}{\partial \bar{x}^\alpha} \bar{A}^\alpha = \delta_j^k A^j = A^k$$

or

$$\boxed{A^i = \frac{\partial x^i}{\partial \bar{x}^\alpha} \bar{A}^\alpha} \quad \text{--- (2)}$$

Covariant vector:

A set of N quantities A_i which are functions of the N coordinates x^i are said to be the components of a covariant vector of they transform given as

$$\bar{A}_\alpha = \frac{\partial x^j}{\partial \bar{x}^\alpha} A_j \quad \text{——— (3)}$$

under a change of coordinates from x^i to $\bar{x}^\alpha$, where $\bar{A}_\alpha$ are the components of the vector in the barred coordinate system. Inverse transformation is given by

$$A_i = \frac{\partial \bar{x}^\alpha}{\partial x^i} \bar{A}_\alpha \quad \text{——— (4)}$$

Q: Show that velocity and acceleration are contravariant vectors and that the gradient of a scalar field is a covariant vector.

Soln.

We have already obtain relations (see earlier class notes on tensor) given by

$$d\bar{x}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^i} dx^i \quad \text{——— (5)}$$

and $$dx^i = \frac{\partial x^i}{\partial \bar{x}^\alpha} d\bar{x}^\alpha \quad \text{— (8)}$$

(i) Let t denotes time. Dividing Eq (8) by dt

$$\frac{\partial \bar{x}^\alpha}{dt} = \frac{\partial \bar{x}^\alpha}{\partial x^i}\frac{dx^i}{dt} \quad \text{—} (7)$$

Next, we define velocity components in barred and unbarred coordinat systems given by

$$\bar{v}^\alpha = \frac{d\bar{x}^\alpha}{dt}, \quad v^i = \frac{dx^i}{dt}.$$

Now Eq. (7) can be written as

$$\boxed{\bar{v}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^i} v^i} \quad \text{—} (8)$$

See Eq. (1) and compare with (8). v^i is a contravariant vector. If you take derivative of (8) again we get,

$$\frac{d\bar{v}^\alpha}{dt} = \frac{\partial \bar{x}^\alpha}{\partial x^i}\frac{dv^i}{dt}$$

or

$$\boxed{\bar{a}^\alpha = \frac{\partial \bar{x}^\alpha}{\partial x^i} a^i} \quad \text{—} (9) \quad \{ a \to \text{acceleration}$$

acceleration is also a contravariant vector.

(ii) Now consider $\phi \equiv \phi(x^i)$ be a scalar field. Thus, its ~~……~~ functions form will remain same under coordinate transformation

$$\phi(x^i) = \bar{\phi}(\bar{x}^\alpha) = \phi(\bar{x}^\alpha)$$

Gradient of the scalar field will be a vector whose components can be defined by

$$A_i = \frac{\partial \phi}{\partial x^i}, \quad \bar{A}_\alpha = \frac{\partial \bar{\phi}}{\partial \bar{x}^\alpha} = \frac{\partial \phi}{\partial \bar{x}^\alpha}$$

Next, using partial derivative, we can write

$$\frac{\partial \phi}{\partial x^i} = \frac{\partial \phi}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^\alpha}{\partial x^i}$$

or $$A_i = \frac{\partial \bar{x}^\alpha}{\partial x^i} \bar{A}_\alpha \quad \text{——} \quad (10)$$

See Eq. (4). and Eq. (10), It is clear that gradient of a scalar field is a covariant vector.

Tensor of Second Rank :

A set of N^2 functions A^{ij} are said to be the components of a contravariant tensor of rank two if they transform according to

$$\bar{A}^{\alpha\beta} = \frac{\partial \bar{x}^\alpha}{\partial x^i} \frac{\partial \bar{x}^\beta}{\partial x^j} A^{ij} \quad \text{——} \quad (11)$$

under coordinate transformations where $\bar{A}^{\alpha\beta}$ are the components of the tensor in barred coordinate system.

Similarly for covariant tensor of second rank, we have

$$\bar{A}_{\alpha\beta} = \frac{\partial x^i}{\partial \bar{x}^\alpha} \frac{\partial x^j}{\partial \bar{x}^\beta} A_{ij}, \quad \text{——} \quad (12)$$

under the coordinate transformation

A set of N^2 functions A^i_j are said to be components of a tensor of contravariant rank one and covariant rank one (or mixed tensor of rank two) if they transform, under coordinate transformation, according to

$$\bar{A}^{\alpha}_{\beta} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial x^{j}}{\partial \bar{x}^{\beta}} A^{i}_{j} \quad (13)$$

A_{ij} (the covariant tensor of rank two) can be represented by a square matrix

$$A_{ij} \equiv \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & & \ddots & \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{bmatrix} \quad (14)$$

General form for a tensor of arbitrary rank:

$$\bar{A}^{\alpha_1 \alpha_2 \cdots \alpha_p}_{\beta_1 \beta_2 \cdots \beta_q} = \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{i_1}} \frac{\partial x^{l_1}}{\partial \bar{x}^{\beta_1}} \cdots \frac{\partial x^{l_q}}{\partial \bar{x}^{\beta_q}} A^{i_1 i_2 \cdots i_p}_{l_1 l_2 \cdots l_q} \quad (15)$$

where $A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ are a set of N^{p+q} functions and are said to be components of a tensor of contravariant rank p and covariant rank q. Total rank $(p+q)$.

and α_r (for $1 \le r \le p$), β_r (for $1 \le r \le q$) are free indices each having value between 1 and N, and i_r, l_s are dummy indices with summation over each from 1 to N.

Equality & null tensor: consider two tensors

$A^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ and $B^{i_1 \cdots i_p}_{j_1 \cdots j_q}$.

If both the tensors have same contravariant and covariant rank and every component of one is equal to the corresponding component of the other, then

$$A^{i_1 i_2 i_3 \cdots i_p}_{j_1 j_2 j_3 \cdots j_q} = B^{i_1 i_2 i_3 \cdots i_p}_{j_1 j_2 j_3 \cdots j_q}. \quad \text{—— (1)}$$

The two tensors have the same contravariant and covariant rank, they are called of the <u>same type</u>

null tensor:- If the N^k components of a tensor of total rank k identically vanish, we say it to be a null tensor.

Addition and substraction of tensor:-

Two tensor of the <u>same type</u> can be added and substracted.

The resultant tensor will have the same rank as the original tensors.

$$C^{i_1 i_2 i_3 \cdots i_p}_{j_1 j_2 j_3 \cdots j_q} = A^{i_1 i_2 i_3 \cdots i_p}_{j_1 j_2 j_3 \cdots j_q} + B^{i_1 i_2 i_3 \cdots i_p}_{j_1 j_2 j_3 \cdots j_q} \quad \text{—— (2)}$$

One can prove that $C^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ is a tensor. Let us write the transformation components of $B^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ in the form

$$\bar{B}^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} = \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial \bar{x}^{\alpha_p}}{\partial x^{i_p}} \frac{\partial x^{l_1}}{\partial \bar{x}^{\beta_1}} \cdots \frac{\partial x^{l_q}}{\partial \bar{x}^{\beta_q}} B^{i_1 \cdots i_p}_{l_1 \cdots l_q} \quad \text{—— (3)}$$

Similarly

$$\bar{A}^{\alpha_1 \alpha_2 \cdots \alpha_p}_{\beta_1 \beta_2 \cdots \beta_q} = \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial \bar{x}^{\alpha_p}}{\partial x^{i_p}} \frac{\partial x^{l_1}}{\partial \bar{x}^{\beta_1}} \cdots \frac{\partial x^{l_q}}{\partial \bar{x}^{\beta_q}} A^{i_1 i_2 \cdots i_p}_{l_1 l_2 \cdots l_q} \quad \text{—— (4)}$$

Adding (3) and (4)

$$\bar{A}^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} + \bar{B}^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} = \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial \bar{x}^{\alpha_p}}{\partial x^{i_p}} \frac{\partial x^{l_1}}{\partial \bar{x}^{\beta_1}} \cdots \frac{\partial x^{l_q}}{\partial \bar{x}^{\beta_q}} \left(A^{i_1 \cdots i_p}_{l_1 \cdots l_q} + B^{i_1 \cdots i_p}_{l_1 \cdots l_q}\right) \quad \text{—— (5)}$$

Now writting the sum of the components of the two tensors in the barred coordinate system as

$$\bar{C}^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} = \bar{A}^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} + \bar{B}^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} . \quad \text{—— (6)}$$

Now from Eq. (5) & (6)

$$\bar{C}^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q} = \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial \bar{x}^{\alpha_p}}{\partial x^{i_p}} \frac{\partial x^{l_1}}{\partial \bar{x}^{\beta_1}} \cdots \frac{\partial x^{l_q}}{\partial \bar{x}^{\beta_q}} C^{i_1 \cdots i_p}_{l_1 \cdots l_q} \quad \text{—— (7)}$$

From Eq. (7) it is obvious that $C^{i_1 \cdots i_p}_{l_1 \cdots l_q}$ is a tensor of contravariant rank p and covariant rank q.

Similary we can define substraction of the two tensors $A^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ and $B^{i_1 \cdots i_p}_{j_1 \cdots j_q}$ ~~let substra~~

$$A^{i_1 \cdots i_p}_{j_1 \cdots j_q} - B^{i_1 \cdots i_p}_{j_1 \cdots j_q} = D^{i_1 \cdots i_p}_{j_1 \cdots j_q} \quad \text{—— (8)}$$

Outer product:

Let A^{ij}_{k} and B^{p}_{q} are two tensors and A^{ij}_{k} has total rank three and, therefore, has N^3 components. B^{p}_{q} has total rank two and N^2 components.

- Each component of one tensor is multiplied by every component of the other. The resulting set of quantities gives a tensor whose rank is the sum of the ranks of the two original tensors.

We write the transformation equations for A^{ij}_{k} and B^{p}_{q} as

$$\bar{A}^{\alpha\beta}_{\gamma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial x^{k}}{\partial \bar{x}^{\gamma}} A^{ij}_{k}, \quad \text{—— (9)}$$

$$\bar{B}^{\rho}_{\sigma} = \frac{\partial \bar{x}^{\rho}}{\partial x^{p}} \frac{\partial x^{q}}{\partial \bar{x}^{\sigma}} B^{p}_{q} \quad \text{—— (10)}$$

Now

$$\bar{A}^{\alpha\beta}_{\gamma} \bar{B}^{\rho}_{\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial x^{k}}{\partial \bar{x}^{\gamma}} \frac{\partial \bar{x}^{\rho}}{\partial x^{p}} \frac{\partial x^{q}}{\partial \bar{x}^{\sigma}} A^{ij}_{k} B^{p}_{q} \quad \text{—— (11)}$$

Let us define $C^{ijp}_{kq} = A^{ij}_{k} B^{p}_{q}$, $\bar{C}^{\alpha\beta\rho}_{\gamma\sigma} = \bar{A}^{\alpha\beta}_{\gamma} \bar{B}^{\rho}_{\sigma}$ —— (12)

- Next, we can write Eq. (11) as

$$\bar{C}^{\alpha\beta\rho}_{\gamma\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial \bar{x}^{\rho}}{\partial x^{p}} \frac{\partial x^{k}}{\partial \bar{x}^{\gamma}} \frac{\partial x^{q}}{\partial \bar{x}^{\sigma}} C^{ijp}_{kq} \quad \text{—— (13)}$$

tensor of contravariant rank 3 and covariant rank 2;

C^{ijp}_{kq} – has total rank 5 and hence N^5 components, each of which ~~is the~~ is the product of one component A^{ij}_{k} with one B^{p}_{q}.

Eq. (13) defines the outer product or Kronecker product of two tensors. This can also be extended to more than two tensors.

Inner product of two tensors: —

Let A^{ij}_k and B^l_m be two tensors. Consider the set of functions $A^{ij}_k B^k_m$ with i, j and m free indices and the index k is summed over from $k = 1, 2, \ldots N$.

There are three free indices in the function $A^{ij}_k B^k_m$, therefore, the number of such functions will be N^3.

In the previous lecture note we have seen than

$$\bar{A}^{\alpha\beta}_{\gamma} \bar{B}^{\rho}_{\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial x^{k}}{\partial \bar{x}^{\gamma}} \frac{\partial \bar{x}^{\rho}}{\partial x^{l}} \frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} A^{ij}_{k} B^{l}_{m} \quad \text{—} (1)$$

for $\rho = \gamma$, in above expression and summing over γ.

$$\bar{A}^{\alpha\beta}_{\gamma} \bar{B}^{\gamma}_{\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial x^{k}}{\partial \bar{x}^{\gamma}} \frac{\partial \bar{x}^{\gamma}}{\partial x^{l}} \frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} A^{ij}_{k} B^{l}_{m} \quad \text{—} (2)$$

Since $\dfrac{\partial x^{k}}{\partial \bar{x}^{\gamma}} \dfrac{\partial \bar{x}^{\gamma}}{\partial x^{l}} = \delta^{k}_{l}$, thus we have

$$\bar{A}^{\alpha\beta}_{\gamma} \bar{B}^{\gamma}_{\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} \delta^{k}_{l} A^{ij}_{k} B^{l}_{m}$$

using the property of delta function, we can write

$$\bar{A}^{\alpha\beta}_{\gamma}\,\bar{B}^{\gamma}_{\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}}\frac{\partial \bar{x}^{\beta}}{\partial x^{j}}\frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} A^{ij}_{k} B^{k}_{m} \quad — (3)$$

Eq. (3) shows that $A^{ij}_{k} B^{k}_{m}$ transform like the component of contravariant rank 2 and covariant rank 1. Let us define

$$\bar{C}^{\alpha\beta}_{\sigma} = \bar{A}^{\alpha\beta}_{\gamma}\,\bar{B}^{\gamma}_{\sigma} \quad — (4)$$

$$\text{and}\quad C^{ij}_{m} = A^{ij}_{k} B^{k}_{m} \quad — (5)$$

In eq (5) C^{ij}_{m} is called as the inner product of two tensors $A^{ij}_{k} B^{k}_{m}$.

<u>H.W.</u> If X^{ij}_{k} and Y^{l}_{m} are two tensors, Show that $X^{ij}_{k} Y^{i}_{m}$ is not a tensor.

Contraction of a tensor:

Let A^{ijk}_{lm} be a tensor, ~~R.~~ rank 5, contravariant rank 3, covariant rank 2, total N^5 component.

for $l = i$, A^{ijk}_{im} will have N^3 components since index i will be summed over.

Let us write transformation equation of A^{ijk}_{lm}

$$\bar{A}^{\alpha\beta\gamma}_{\rho\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}}\frac{\partial \bar{x}^{\beta}}{\partial x^{j}}\frac{\partial \bar{x}^{\gamma}}{\partial x^{k}}\frac{\partial x^{l}}{\partial \bar{x}^{\rho}}\frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} A^{ijk}_{lm}$$

Now taking $\rho = \alpha$, and summing over α

(6).

$$\bar{A}^{\alpha\beta\gamma}_{\alpha\sigma} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{i}} \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial \bar{x}^{\gamma}}{\partial x^{k}} \frac{\partial x^{l}}{\partial \bar{x}^{\alpha}} \frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} A^{ijk}_{lm}$$

$$= \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial \bar{x}^{\gamma}}{\partial x^{k}} \frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} \delta^{l}_{i} A^{ijk}_{lm}$$

or $$\bar{A}^{\alpha\beta\gamma}_{\alpha\sigma} = \frac{\partial \bar{x}^{\beta}}{\partial x^{j}} \frac{\partial \bar{x}^{\gamma}}{\partial x^{k}} \frac{\partial x^{m}}{\partial \bar{x}^{\sigma}} A^{ijk}_{im}$$

$A^{ijk}_{im} \Rightarrow$ Contravariant rank 2
Covariant rank 1

When a tensor is contracted by making one of its covariant index equal to its contravariant index, then the resultant quantity is a tensor whose covariant and contravariant indices are reduced by one, and therefore the total rank is reduced by two. This process is called the contraction of a tensor.

symmetric tensor:

(i) Let us consider a contravariant tensor X^{ij} of rank 2 and if

$$X^{ij} = X^{ji}$$

then $X^{ij} \rightarrow$ symmetric tensor

Similarly for X_{ij}, if $X_{ij} = X_{ji}$

$\rightarrow X_{ij} \rightarrow$ symmetric

(ii) Consider an arbitrary rank tensor, for example

X^{ijk}_{lmn}, if

$$X^{ijk}_{lmn} = X^{jik}_{lmn}$$

then X^{ijk}_{lmn} is symmetric in the first two contravariant indices.

Similarly if, $X^{ijk}_{lmn} = X^{ijk}_{nml}$, then

$X^{ijk}_{lmn} \rightarrow$ symmetric in the first and third covariant indices.

Antisymmetric tensor:

(i) Consider the previous example, i.e., $X^{ij} \rightarrow$ tensor quantity and if

$$X^{ij} = -X^{ji}$$

$\Rightarrow X^{ij}$ is antisymmetric

~~Sim~~ Similarly for X_{ij}, if

$$X_{ij} = -X_{ji}$$

then $X_{ij} \rightarrow$ antisymmetric.

(ii) For tensor X^{ijk}_{lmn} if

$$X^{ijk}_{lmn} = -X^{jik}_{lmn}$$

$X^{ijk}_{lmn} \rightarrow$ antisymmetric in the first two contravariant indices.

Again if $X^{ijk}_{lmn} = -X^{ijk}_{nml}$, then

$X^{ijk}_{lmn} \rightarrow$ antisymmetric in the first and third covariant indices.

Example of symmetric tensor: —

(i) Kronecker delta $\rightarrow \delta_{ij}$ or δ^{ij} etc

(ii) For an isotropic system, stress tensor σ_{ij} or σ^{ij} is symmetric tensor.

Quotient Law:

consider a quantity A and we do not know whether A is tensor or not. If the inner product of A with an arbitrary tensor is itself a tensor then A is also a tensor. This is called the quotient law.

Let $A(i, j, k) \rightarrow$ components of a tensor

Let inner product of ~~(A,~~ $A(i,j,k)$) with an arbitrary tensor B^{pq} is a contravariant tensor of first rank

$$A(i,j,k)\, B^{jk} = C^{i} \quad \text{—— } (1)$$

j,k (Repeated index) are summed over.

Let $\bar{A}(\alpha, \beta, \gamma) \rightarrow N^3$ functions in the barred coordinates and satisfy

$$\bar{A}(\alpha,\beta,\gamma)\, \bar{B}^{\beta\gamma} = \bar{C}^{\alpha} \quad \text{—— } (2)$$

to write $\bar{B}$ and $\bar{C}$ in terms of unbarred components, we write the above eq. as

$$\bar{A}(\alpha,\beta,\gamma)\frac{\partial \bar{x}^\beta}{\partial x^j}\frac{\partial \bar{x}^\gamma}{\partial x^k}B^{jk} = \frac{\partial \bar{x}^\alpha}{\partial x^i}C^i$$

$$= \frac{\partial \bar{x}^\alpha}{\partial x^i}A(i,j,k)B^{jk}$$

or $\left[\bar{A}(\alpha,\beta,\gamma)\frac{\partial \bar{x}^\beta}{\partial x^j}\frac{\partial \bar{x}^\gamma}{\partial x^k} - \frac{\partial \bar{x}^\alpha}{\partial x^i}A(i,j,k)\right]B^{jk} = 0$ (from eq. 2) —— (3)

This is true for ~~or~~ an arbitrary tensor B^{jk}. Therefore,

$$\bar{A}(\alpha,\beta,\gamma)\frac{\partial \bar{x}^\beta}{\partial x^j}\frac{\partial \bar{x}^\gamma}{\partial x^k} = \frac{\partial \bar{x}^\alpha}{\partial x^i}A(i,j,k)$$

Now multiplying $\frac{\partial x^j}{\partial \bar{x}^\rho}\frac{\partial x^k}{\partial \bar{x}^\sigma}$ (inner multiplication) on both sides, we obtain

$$\bar{A}(\alpha,\beta,\gamma)\,\delta_{\beta\rho}\,\delta_{\gamma\sigma} = \frac{\partial \bar{x}^\alpha}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^\rho}\frac{\partial x^k}{\partial \bar{x}^\sigma}A(i,j,k)$$

$$\boxed{\bar{A}(\alpha,\rho,\sigma) = \frac{\partial \bar{x}^\alpha}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^\rho}\frac{\partial x^k}{\partial \bar{x}^\sigma}A(i,j,k)} \quad \text{—— (4)}$$

From eq. (4), we see that $A(i,j,k)$ is a tensor of contravariant rank one and covariant rank 2. Now writting Eq. (4) in tensor notation

$$\bar{A}^\alpha_{\rho\sigma} = \frac{\partial \bar{x}^\alpha}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^\rho}\frac{\partial x^k}{\partial \bar{x}^\sigma}A^i_{jk} \quad \text{—— (5)}$$

<u>Conjugate or reciprocal tensors</u>:

Let A_{ij} → Symmetric covariant tensor of rank 2, such that when A_{ij} is expressed as a matrix then $\det(A_{ij}) = |A_{ij}| \neq 0$. Now define

$$B^{ij} = \frac{\text{cofactor of } A_{ij}}{|A_{ij}|}$$

Then B^{ij} is a symmetric contravariant tensor of rank 2 called as the conjugate tensor of A_{ij} and

$$\boxed{B^{ij}A_{kj} = \delta^i_k}$$

Matrices:

A matrix of order m by n is an array of quantities A_{ij} called elements arranged in m rows and n columns.

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \text{ or } \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix}$$

It may be denoted by $[A_{ij}]$ or (A_{ij}) for $i = 1, 2, \cdots m$; $j = 1, 2, \cdots, n$.

* for $m = n$, matrix is square matrix of order m by m
* for $m = 1$ $\rightarrow$ Row matrix or Row vector
* for $n = 1$ $\rightarrow$ Column matrix or Column vector

* For the square matrix $(m = n)$. the diagonal containing elements $A_{11}, A_{22}, A_{33}, \cdots A_{mm}$ is called the principal or main diagonal

Unit Matrix (I) $\rightarrow$ $A_{ii} = 1$ and $A_{ij} = 0$ for $i \neq j$
(For square matrix only)

metric tensor:

In rectangular coordinate the differential arc length is given by

$$ds^2 = dx^2 + dy^2 + dz^2 \quad \text{--- (1)}$$

The transformation of the above arc length to general curvilinear coordinates (discussed in earlier lecture notes) we obtain

$$ds^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij}\, du_i\, du_j \quad \text{--- (2)}$$

For N-dimensional space having coordinates $(x^1, x^2, \ldots x^N)$ the line element ds given in quadratic form is called metric form or metric

$$ds^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij}\, dx^i dx^j$$

or $$\boxed{ds^2 = g_{ij}\, dx^i dx^j} \quad \text{--- (3)}$$

where we have used Einstein summation convention. (Repeated indices are summed over).

$g_{ij} \rightarrow$ may be functions of x^i with the condition that $\det(g_{ij}) \neq 0$ and the space is called as Riemannian space. For the particular case that g_{ij} are independent of $x^i \rightarrow$ space becomes Euclidean space.

Associated tensors:

Assume $A^i \rightarrow$ arbitrary contravariant vector

Now $$A^i g_{ij} = A_j$$

A^i and $A_j \rightarrow$ contravariant and covariant components of same vector

Again

$$A_v g^{jk} = A^i g_{ij} g^{jk} = A^i \delta_i^k = A^k \quad \text{— (4)}$$

Eq. (4) shows that A_j & A^i have reciprocal relation.

Tensors A^i and A_i are called associate tensors.

Moment of inertia tensor:

The angular momentum $\vec{L} = \sum_i m_i (\vec{r}_i \times \vec{v}_i)$ — (5)

$m_i \to$ mass of the i-th point mass
$\vec{r}_i \to$ position vector of m_i w.r.t. origin
$\vec{v}_i \to$ linear velocity of m_i

If $\vec{\omega} \to$ angular velocity vector of the body

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i$$

$$\therefore \vec{L} = \sum m_i [\vec{r}_i \times (\omega \times \vec{r}_i)] = \sum_i m_i [\omega r_i^2 - \vec{r}_i (\vec{r}_i \cdot \vec{\omega})] \quad \text{— (6)}$$

{we have used $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Now writting in cartesian components

$$\vec{L} = (L_x, L_y, L_z), \quad \vec{r}_i = (x_i, y_i, z_i)$$
$$\vec{\omega} = (\omega_x, \omega_y, \omega_z),$$

Now from eq (6)

$$\begin{aligned} L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ L_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{aligned} \quad \text{— (7)}$$

where, $I_{xx} = \sum_i m_i (r_i^2 - x_i^2)$, $I_{yy} = \sum_i m_i (r_i^2 - y_i^2)$, $I_{zz} = \sum_i m_i (r_i^2 - z_i^2)$

$I_{xy} = I_{yx} = -\sum_i m_i x_i y_i$, $I_{yz} = I_{zy} = -\sum_i m_i y_i z_i$

$I_{zx} = I_{xz} = -\sum_i m_i z_i x_i$

Eq. (7) show that $\vec{L}$ is not parallel to $\vec{\omega}$. We write

$\boxed{L_i = I_{ij}\omega_j}$ — (8), L, ω } $\to$ vectors

$I_{ij} \to$ tensor (from quotient law it is clear from eq. 8)